Notes on random walk dynamics of line digraphs

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January 17, 2024

Abstract

We argue that for a strongly connected simple directed graph G, random walk on vertices of G and $L(G)$ exhibit identical dynamics.

0 Introduction

Despite Whitney graph isomorphism theorem, visitation frequencies generated by a random walk on vertices might approximate to different values for edges of G and vertices of its line graph, $L(G)$ (Evans and Lambiotte, 2009).

One can show that (see Blum, Theorem 5.2) the transition matrix of a strongly connected simple directed graph has 1 as a left eigenvalue. Given G, its adjacency matrix $A(G)$ and its transition matrix $P(G)$, there is a unique π, called stationary distribution, such that $\pi P(G) = \pi$ and $\sum_{i=1} \pi_i = 1$. This is also the case for a connected simple undirected graph.

All vectors are $1 \times k$ shaped matrices.

1 Undirected case

In this section G is a connected simple undirected graph with n vertices and m edges. Fix any order on $V(G)$ and $E(G)$. Let π be its stationary distribution.

Let $\deg(v_i)$ be the degree of the vertex v_i and

$$S = \operatorname{diag}(\frac{1}{\deg(v_1)}, \ldots, \frac{1}{\deg(v_n)})$$

be the diagonal matrix containing individual transition probabilities.

Let $\mathbf{1}_k$ denote the constant vector of length k with all elements equal to 1. We have $SA(G) = P(G)$ and $\pi_i = \frac{\deg(v_i)}{2m}$. Furthermore, $\pi S = \dfrac{1}{2m}\mathbf{1}_n$.

Definition 1. The incidence matrix of G is $B(G)$, where

$$B(G)_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ is indicent with } e_j \\ 0, & \text{otherwise} \end{cases}$$

Observe that $B(G)$ has exactly two 1s in each column, so we have

$$\frac{1}{2m}\mathbf{1}_n B(G) = \frac{1}{m}\mathbf{1}_m$$

One can interpret $\frac{1}{m}\mathbf{1}_m$ as the edge visitation frequencies under a random walk on vertices and π as vertex visitation frequencies. For an edge $e_{i,j}$, we are expecting a visit with

$$\frac{\pi_i}{\deg(v_i)} + \frac{\pi_j}{\deg(v_j)} = \frac{1}{m}$$

probability.

However, $\pi_{L(G)}$, stationary distribution of $L(G)$ won't be equal to $\dfrac{1}{m}\mathbf{1}_m$ unless G is a regular graph.

Intuitively, this difference in the dynamics of visitations comes from the observation that, in G, one can go back and forth on an edge, which should correspond to "staying still" or looping in $L(G)$, which is not a valid next step.

2 Directed case

In this section G is a strongly connected simple directed graph with n vertices and m edges. Fix any order on $V(G)$ and $E(G)$. Let π be its stationary distribution.

Let $\deg^{\text{out}}(v_i)$ be the out degree of the vertex v_i, namely the number of edges which have v_i at its head. There is no closed form solution for π as it was in the case for undirected. However, given π, the edge $e_{i,j}$ will have the visitation frequency of

$$\frac{\pi_i}{\deg^{\text{out}}(v_i)} \tag{1}$$

Definition 2. The in-incidence matrix of G is $B_I(G)$, where

$$B_I(G)_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ is the head of } e_j \\ 0, & \text{otherwise} \end{cases}$$

Definition 3. The out-incidence matrix of G is $B_O(G)$, where

$$B_O(G)_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ is the tail of } e_j \\ 0, & \text{otherwise} \end{cases}$$

Lemma 1. We have $A(G) = B_I B_O^T$ and $A(L(G)) = B_O^T B_I$. Furthermore, let

$$S = \text{diag}(\frac{1}{\deg^{\text{out}}(v_1)}, \dots, \frac{1}{\deg^{\text{out}}(v_n)})$$

Then we have

$$P(G) = SA(G)$$

and

$$P(L(G)) = B_O^T S B_I$$

Proof. By the definition of $P(\cdot)$, we have

$$P(L(G))_{e_{i,j},e_{p,q}} = \begin{cases} \frac{1}{\deg^{\text{out}}(e_{i,j})}, & \text{if } j = p \\ 0, & \text{otherwise} \end{cases}$$

An edge in $L(G)$ goes out of $e_{i,j}$ if there was an edge in G starting from v_j. Hence we have $\deg^{\text{out}}(e_{i,j}) = \deg^{\text{out}}(v_j) = \frac{1}{S_{j,j}}$. Also,

$$\begin{aligned} B_O^T(SB_I)_{e_{i,j},e_{p,q}} &= \sum_{k=1}^{n} (B_O^T)_{e_{i,j},k}(SB_I)_{k,e_{p,q}} \\ &= \sum_{k=1}^{n} (B_O)_{k,e_{i,j}}(SB_I)_{k,e_{p,q}} \\ &= \sum_{k=1}^{n} (B_O)_{k,e_{i,j}} S_{k,k}(B_I)_{k,e_{p,q}} \end{aligned}$$

Now observe that $(B_O)_{k,e_{i,j}}$ is 1 if $k = j$ and 0 otherwise. Similarly, $(B_I)_{k,e_{p,q}}$ is 1 if $k = p$ and 0 otherwise. Hence, we get $S_{j,j}$ if $k = j$, $p = j$ and 0 otherwise, which is what we wanted to show.

Observe that this calculation was independent of the order on $V(G)$ and $E(G)$ that was fixed in the beginning.

Other parts of the lemma follows with a similar calculation.

□

Edge visitation frequencies on G given a random walk on vertices should be

$$F = \pi S B_I$$

To see this, observe that B_I has only one 1 on each column. Given an edge $e_{i,j}$, B_I has that single 1 on column corresponding to $e_{i,j}$ (which is probably not the column j, mind the notation) exactly at the row i. So we have

$$F_{e_{i,j}} = (\pi S)_{i,i} = \pi_i S_{i,i} = \frac{\pi_i}{\deg^{\text{out}}(v_i)}$$

as we wanted in equation 1.

Lemma 2. F is the stationary distribution for $L(G)$, i.e.

$$FP(L(G)) = F$$

Proof. We have $\pi P(G) = \pi$ and via Lemma 1

$$P(L(G)) = B_O^T S B_I$$

Furthermore

$$\pi = \pi SA(G) = \pi S B_I B_O^T$$

Multiplying both sides with SB_I, we have

$$\begin{aligned} F = \pi S B_I &= \pi S B_I B_O^T S B_I \\ &= \pi S B_I P(L(G)) \\ &= FP(L(G)) \end{aligned}$$

□

With this, we can say that random walk on G gives the same edge visitation frequencies as random walk on $L(G)$ gives for the vertices. Hence, there is no difference on dynamics of G and $L(G)$ in this sense, as opposed to the case of undirected.

Intuitively, this is obvious. Any step on G (or $L(G)$) has a clear corresponding step in $L(G)$ (or G).

References

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