This is an excerpt from my Bachelor's thesis, showing only the first chapter.

Dold-Kan Correspondence

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Contents

Introduction

Recall the construction of singular homology of a topological space.

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, "Sing"] \arrow[d, "H_n^{Sing}"] & \mathbf{sSet} \arrow[r, "\mathbb{Z}"] & \mathbf{sAb} \arrow[d, "M"] \\
\mathbf{Ab} & & Ch_\bullet^+(\mathbf{Ab}) \arrow[ll, "H_n"']
\end{tikzcd}$$

The main goal of this thesis is to focus on the passage from simplicial abelian groups to chain complexes.

A simplicial set is like a combinatorial space built up out of gluing abstract simplices to each other. Simplicial sets are purely algebraic and do not carry any actual topology, but provide a suitable environment for homotopy theory.

The Dold-Kan Correspondence shows that the category of connective chain complexes of abelian groups and simplicial abelian groups are equivalent categories observed by the functors

$$N : \mathbf{sAb} \longrightarrow Ch_\bullet^+(\mathbf{Ab}) \text{ and } \Gamma : Ch_\bullet^+(\mathbf{Ab}) \longrightarrow \mathbf{sAb}.$$

Moreover, there's a natural isomorphism between the homology groups of chain complexes and and simplicial homotopy groups of simplicial abelian groups.

This thesis starts with providing some preliminary information about these categories in section 1. In section 2, we give a careful and detailed proof of the correspondence and show that the normalized chain complex and the Moore complex are chain homotopic. Combinatorial details of this last theorem are particularly original. Then we end in section 3 by discussing the definition of simplicial homotopy and giving an application of the correspondence through Eilenberg-Maclane spaces.

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1 The Dold-Kan Correspondence

The main goal of this chapter is to prove that the chain complexes and simplicial abelian groups are equivalent categories and the normalized chain complex carries all the necessary data for homology.

This correspondence is (independently) due to Albrecht Dold and Daniel Kan (1958). Historical references for the Dold–Kan correspondence are [1] and [7].

Majority of the content in this section will be very combinatorial. Throughout this chapter, one might want to keep a close eye on the Table 1. Probably the key theorem is Theorem 1.5, which implies why the "normalized" is used for the normalized chain complex. Observing this splitting for the singular simplicial set for 2-simplices. We suggest looking at the n-simplices as n-cells and applying face maps to as taking boundary.

1.1 Categorical equivalence

Throughout this chapter A is always a simplicial abelian group and $[n] \in \Delta$ is replaced by n to ease the notation to for eye and this abuse of notation should be no problem thanks to the context.

Lemma 1.1. *The normalized chain complex NA associated to the simplicial abelian group A, given by the groups*

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i) \subset A_n$$

and by the maps

$$NA_n \xrightarrow{(-1)^n d_n} NA_{n-1}.$$

is a subcomplex of MA.

Proof. Firstly, observe that for $i < n$ the map $NA_n \xrightarrow{d_i} NA_{n-1}$ is the zero map by definition. Additionally, $NA_n \xrightarrow{d_n} NA_{n-1}$ is well-defined, because for $j < n-1$, by the simplicial identity S1 we have $d_j d_n = d_{n-1} d_j$. Lastly, we have $NA_0 = MA$ by definition. □

This construction defines a functor $N : \mathbf{sAb} \longrightarrow \mathbf{Ch}$, where given a simplicial map $f_n : A_n \longrightarrow B_n$, $N(f)_n$ is defined as simply restricting the domain to NA_n. This gives a chain map in return, while the differential is just sum of simplicial structure maps, which by definition commute with the map f. We can also make the exact same point about the construction of a Moore complex.

Lemma 1.2. *Sum of degenerate simplices in degree n*

$$DA_n = \langle s_i(A_{n-1}) \rangle_{i=0}^{n-1} \subset A_n$$

gives a subcomplex of MA.

Proof. We take the same differential, so we only need to show that $DA_n \xrightarrow{\partial_n} DA_{n-1}$ is well-defined. It's enough to show that for an $s_i a \in DA_n$, we have $\partial_n(s_i a) \in DA_{n-1}$. If we open up the sum, $\partial_n(s_i a) = \sum_{j=0}^{n}(-1)^j d_j s_i a = \sum_{j=i,i+1}(-1)^j d_j s_i a + \sum_{j<i}(-1)^j d_j s_i a + \sum_{j>i+1}(-1)^j d_j s_i a$. By the simplicial identities S3, S2 and S4 we have $(-1)^i(d_i s_i a - d_{i+1} s_i a) = 0$, $d_j s_i = s_{i-1} d_j$ for $j < i$ and $d_j s_i = s_i d_{j-1}$ for $j > i+1$. This implies that $\partial_n(s_i a)$ can be written as the sum of degenerate simplices. Lastly, observe that in the calculation d_{i+1} is always well-defined, $DA_0 = 0$ and the case $n = 1$ is naturally included. □

Lemma 1.3. *For every $k > 0$,*

$$N_k A_n = \begin{cases} \bigcap_{i=0}^{k-1} ker(d_i) & \text{for } n \geq k+1, \\ NA_n & \text{for } n \leq k \end{cases}$$

is a subcomplex of MA.

Remark. *See Table 1. Observe that we have $N_0 A = MA$ and $N_{k+1}A_n \subset N_k A_n$. Also Table 1 "converges" to NA, i.e., $\bigcap_{k\geq 0} N_k A = NA$.*

Proof. Observe Table 1. For $n \leq k$, the statement is proven in Lemma 1.1. Starting from $n \geq k+1$, we want to show that the map

$$\bigcap_{i=0}^{k-1} \ker(d_i) = N_k A_n \subset A_n \xrightarrow{\partial_n} \bigcap_{i=0}^{k-1} \ker(d_i) = N_k A_{n-1} \subset A_{n-1}$$

is well-defined. Let $i < k$, then

$$\begin{aligned} d_i \partial_n(x) &= d_i(\sum_{j=0}^{n}(-1)^j d_j x) = \sum_{j=0}^{n}(-1)^j d_i d_j x \\ &= \sum_{j=k}^{n}(-1)^j d_i d_j x = \sum_{j=k}^{n}(-1)^j d_{j-1} d_i x = 0 \end{aligned}$$

where as usual we used a simplicial identity S1. □

Lemma 1.4. *For every $k > 0$,*

$$D_k A_n = \begin{cases} \langle s_i(A_{n-1}) \rangle_{i=0}^{k-1} & \text{for } n \geq k+1, \\ DA_n & \text{for } n \leq k \end{cases}$$

is a subcomplex of MA.

Table 1: Filtration of the Normalized Chain Complex

$\xrightarrow{n}$

$\downarrow k$

N_0A_0	N_0A_1	N_0A_2	N_0A_3	N_0A_4	N_0A_5	N_0A_6	N_0A_7
$A_0 = NA_0$	A_1	A_2	A_3	A_4	A_5	A_6	A_7
N_1A_0	N_1A_1	N_1A_2	N_1A_3	N_1A_4	N_1A_5	N_1A_6	N_1A_7
$A_0 = NA_0$	NA_1	$\bigcap_{i=0}^{0} \ker(d_i)$	$\bigcap_{i=0}^{0} \ker(d_i)$	$\bigcap_{i=0}^{0} \ker(d_i)$	$\bigcap_{i=0}^{0} \ker(d_i)$	$\bigcap_{i=0}^{0} \ker(d_i)$	$\bigcap_{i=0}^{0} \ker(d_i)$
N_2A_0	N_2A_1	N_2A_2	N_2A_3	N_2A_4	N_2A_5	N_2A_6	N_2A_7
$A_0 = NA_0$	NA_1	NA_2	$\bigcap_{i=0}^{1} \ker(d_i)$	$\bigcap_{i=0}^{1} \ker(d_i)$	$\bigcap_{i=0}^{1} \ker(d_i)$	$\bigcap_{i=0}^{1} \ker(d_i)$	$\bigcap_{i=0}^{1} \ker(d_i)$
N_3A_0	N_3A_1	N_3A_2	N_3A_3	N_3A_4	N_3A_5	N_3A_6	N_3A_7
$A_0 = NA_0$	NA_1	NA_2	NA_3	$\bigcap_{i=0}^{2} \ker(d_i)$	$\bigcap_{i=0}^{2} \ker(d_i)$	$\bigcap_{i=0}^{2} \ker(d_i)$	$\bigcap_{i=0}^{2} \ker(d_i)$
N_4A_0	N_4A_1	N_4A_2	N_4A_3	N_4A_4	N_4A_5	N_4A_6	N_4A_7
$A_0 = NA_0$	NA_1	NA_2	NA_3	NA_4	$\bigcap_{i=0}^{3} \ker(d_i)$	$\bigcap_{i=0}^{3} \ker(d_i)$	$\bigcap_{i=0}^{3} \ker(d_i)$
N_5A_0	N_5A_1	N_5A_2	N_5A_3	N_5A_4	N_5A_5	N_5A_6	N_5A_7
$A_0 = NA_0$	NA_1	NA_2	NA_3	NA_4	NA_5	$\bigcap_{i=0}^{4} \ker(d_i)$	$\bigcap_{i=0}^{4} \ker(d_i)$
N_6A_0	N_6A_1	N_6A_2	N_6A_3	N_6A_4	N_6A_5	N_6A_6	N_6A_7
$A_0 = NA_0$	NA_1	NA_2	NA_3	NA_4	NA_5	NA_6	$\bigcap_{i=0}^{5} \ker(d_i)$

Differentials are going in the left direction. Each row is a subcomplex of MA.

Proof. Observe Table 1. For $n \leq k$, the statement is proven in Lemma 1.2. Starting from $n \geq k+1$, we want to show that the map

$$\langle s_i(A_{n-1})\rangle_{i=0}^{k-1} = D_k A_n \subset A_n \xrightarrow{\partial_n} \langle s_i(A_{n-1})\rangle_{i=0}^{k-1} = D_k A_{n-1} \subset A_{n-1}$$

is well-defined. A quick look at the proof of Lemma 1.2 makes it clear that, we use only s_i and s_{i-1} to write the sum, which proves this lemma. $\square$

Remark. *Observe that by definition we have $D_k A_n \subset D_{k+1} A_n$, $D_0 A = 0$ and $\bigcup_{k \geq 0} D_k A = DA$. Theorem 1.5 shows that $D_k A_n$ fully complements Table 1 to MA.*

Theorem 1.5. *The natural inclusion map*

$$NA_n \oplus DA_n \xrightarrow{(i,i)} A_n$$

is an isomorphism of abelian groups.

Corollary 1.6. *The map*

$$NA \xrightarrow{p \circ i} MA/DA$$

is an isomorphism of chain complexes.

Proof of the Corollary 1.6. We have already seen that each complex has the same differential. Furthermore $p \circ i$ is a composition of chain maps. Hence, Theorem 1.5 directly gives the isomorphism. For the reverse direction, one needs to show that the corresponding exact sequence splits. This can be seen in the proof of Theorem 1.5. $\square$

Proof of the Theorem 1.5. Naturality of the inclusion map is self-evident.

We want to show that for every k and n,

$$N_k A_n \oplus D_k A_n \xrightarrow[(i,i)]{\simeq} A_n. \tag{1}$$

which would give the result that we want for $k = n$.

For $k = 0$ we have $N_0 A_n = A_n$ and $D_0 A_n = 0$ and for $n = 0$ we have $N_k A_0 = A_0$ and $D_k A_0 = 0$, for which the statement is obvious. Now, let $k > 0$, $n > 0$. A quick look at the Table 1 makes it clear that, it's enough to show (1) for $k \leq n$. For $n = k$ we have the statement of the theorem. We will do induction on k. We assume $1 \leq k \leq n$.

For $k = 1$, we want to show that $N_1 A_n \oplus D_1 A_n = \ker(d_0) \oplus \mathrm{im}(s_0) = A_n$. For this, let $x \in A_n$ such that $d_0 x = 0$ and there exists a $y \in A_{n-1}$ such that $s_0 y = x$. But then we have $0 = d_0 x = d_0 s_0 y = y$ by the simplicial identity S3. This shows that the subgroups have trivial intersection. Secondly, $x - s_0 d_0 x$ is in the kernel of d_0, because: $d_0 x - d_0 s_0 d_0 x = d_0 x - d_0 x = 0$. Also observe that d_0 is a section for s_0. The upcoming split exact sequence is a generalization of this situation.

Now we assume that we have $A_n = N_{k-1}A_n \oplus D_{k-1}A_n$ and $A_{n-1} = N_{k-1}A_{n-1} \oplus D_{k-1}A_{n-1}$. First, we have the following split exact sequence (it splits thanks to the S3)

$$0 \longrightarrow N_{k-1}A_{n-1} \xrightarrow{s_{k-1}} N_{k-1}A_n \xrightarrow{\mathrm{id}-s_{k-1}d_{k-1}} N_kA_n \longrightarrow 0$$

First of all, it's well-defined. Let $x \in N_{k-1}A_{n-1}$. Then for $j < k-1$ by the simplicial identity S1 we have $d_j s_{k-1}x = s_{k-2}d_{j-1}x = 0$. Second map is covered below. By the splitting lemma, showing that we have split exact sequence is same as showing that

$$N_kA_n \oplus N_{k-1}A_{n-1} \xrightarrow{(i,s_{k-1})} N_{k-1}A_n$$

is an isomorphism. To show that it's injective, assume that $x \in N_kA_n$ and $y \in N_{k-1}A_{n-1}$ and $x + s_{k-1}y = 0$. Then $x = -s_{k-1}y$ and by the simplicial identity S3 $y = d_{k-1}s_{k-1}y = -d_{k-1}x = 0$. To show that it's surjective, observe that by S1 and S2, for $j < k-1$ and $x \in N_{k-1}A_n$ we have

$$d_j(x - s_{k-1}d_{k-1}x) = d_jx - d_js_{k-1}d_{k-1}x = s_{k-2}d_jd_{k-1} = s_{k-2}d_{k-2}d_jx = 0$$

and for $j = k-1$ we have

$$d_{k-1}(x - s_{k-1}d_{k-1}x) = d_{k-1}x - d_{k-1}s_{k-1}d_{k-1}x = d_{k-1}x - d_{k-1}x = 0$$

so $x - s_{k-1}d_{k-1}x \in N_kA_n$ (so the split exact sequence is well-defined). As we shown before $d_{k-1}x \in N_{k-1}A_{n-1}$ and then obviously $(x - s_{k-1}d_{k-1}x, d_{k-1}x)$ hits x. Similarly we have the following split exact sequence

$$0 \longrightarrow A_{n-1}/D_{k-1}A_{n-1} \xrightarrow{s_{k-1}} A_n/D_{k-1}A_n \longrightarrow A_n/D_kA_n \longrightarrow 0$$

Let $x \in D_{k-1}A_{n-1}$ and without loss of generality, for $j < k-1$, $x = s_jy$ for a $y \in A_{n-2}$. Then by the simplicial identity S5 we have $s_{k-1}x = s_{k-1}s_jy = s_js_{k-2}y$, which means that $s_{k-1}x \in D_{k-1}A_n$. The fact that it's exact is obvious, because the second map is just projection under s_{k-1}.

Finally we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & N_{k-1}A_{n-1} \arrow[r, "s_{k-1}"] \arrow[d, "\simeq"] & N_{k-1}A_n \arrow[r, "\mathrm{id}-s_{k-1}d_{k-1}"] \arrow[d, "\simeq"] & N_kA_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1}/D_{k-1}A_{n-1} \arrow[r, "s_{k-1}"] & A_n/D_{k-1}A_n \arrow[r] & A_n/D_kA_n \arrow[r] & 0
\end{tikzcd}$$

It's completely trivial to see that the first square commutes, we have natural inclusions, projections and observe that s_{k-1} in the second exact sequence is defined to commute with the projection. For the second square, only additional detail is that $s_{k-1}d_{k-1}x$ gets killed under the projection. Here we also see again how the projection map in Corollary 1.6 comes into play. $\square$

Now we continue by defining the functor $\Gamma : \mathbf{Ch} \longrightarrow \mathbf{sAb}$ which is going to be inverse to N. Let C be a chain complex and define

$$\Gamma(C)_n = \bigoplus_{n \twoheadrightarrow k} C_k$$

where we have the direct sum indexed by the surjective maps between n and k in Δ, so we have abelian groups for each n. Now we need to define the simplicial structure maps. Let $\theta : m \longrightarrow n$ be a map in Δ^{op}. In order to define the map

$$\bigoplus_{n \twoheadrightarrow k} C_k \longrightarrow \bigoplus_{m \twoheadrightarrow l} C_l$$

it's enough to define what happens to a summand, say C_k with the index $\sigma : n \twoheadrightarrow k$. Let $m \overset{\sigma'}{\twoheadrightarrow} s \overset{\phi}{\hookrightarrow} k$ be the unique epi-mono factorization (Lemma 0.3) of $m \xrightarrow{\theta} n \overset{\sigma}{\twoheadrightarrow} k$. What we want to do here is to first define $C_k \xrightarrow{\phi^*} C_s$ and put the result in the direct sum with the index $\sigma' : m \twoheadrightarrow s$. Here we get the motivation from Theorem 1.1. Observe that if we had NA_k and NA_s instead of C_k and C_s, then a monomorphism $\phi : s \hookrightarrow k$ would always induce the zero map, unless we have $d^n : n-1 \hookrightarrow n$ or $\mathrm{id} : n \hookrightarrow n$. In order to have a functor, we need to define the identity as identity map, however for the case of $C_n \xrightarrow{d_n} C_{n-1}$, let the map be the differential of C, so $(d^n)^* = \partial_n$ and for all other cases be the zero map. Observe that we have $\mathrm{id}^* = \mathrm{id}$.

Lemma 1.7. $\Gamma : \mathbf{Ch} \longrightarrow \mathbf{sAb}$ *as described above defines a functor.*

Proof. First we need to show that the definition above gives a simplicial abelian group. Only missing part for this is to show that given $\theta_1 : n_1 \longrightarrow n_2$ and $\theta_1 : n_2 \longrightarrow n_3$, we have ${\theta_1}^*{\theta_2}^* = (\theta_2\theta_1)^*$. For this we want to look what happens to a summand of $\Gamma(C)_{n_3}$. Let C_{k_3} with the index $\sigma_3 : n_3 \twoheadrightarrow k_3$ be the summand. Look at the following commutative diagram:

$$\begin{tikzcd}
n_1 \arrow[r, "\theta_1"] \arrow[d, two heads, "\sigma_1"] & n_2 \arrow[r, "\theta_2"] \arrow[d, two heads, "\sigma_2"] & n_3 \arrow[d, "\sigma_3"] \\
k_1 \arrow[r, hook, "\phi_1"] & k_2 \arrow[r, hook, "\phi_2"] & k_3
\end{tikzcd}$$

Let's assume that the second square is the epi-mono factorization of $n_2 \xrightarrow{\theta_2} n_3 \overset{\sigma_3}{\twoheadrightarrow} k_3$ and that first square is also the epi-mono factorization. Observe that σ_1 and $\phi_2\phi_1$ give the epi-mono factorization of the larger square too. This way the question is naturally reduced to the monomorphisms. Here nearly all cases get reduced to the zero map or trivial because of identity. The only interesting case is when we have $n-2 \xhookrightarrow{d^{n-2}} n-1 \xhookrightarrow{d^{n-1}} n$ But $d^{n-2*} = \partial_{n-1}$ and $d^{n-1*} = \partial_n$, so we have $\partial_{n-1}\partial_n = 0$ (because we have a chain complex) and $(d^{n-2}d^{n-1})^* = 0$ (because of the definition of the functor).

Now that we know what happens to the objects, let's look at the morphisms, which have as usual a very plain definition. Let $C \xrightarrow{f} D$ be a chain map. $\Gamma(f) : \Gamma(C) \longrightarrow \Gamma(D)$ is defined on the summand A_n with the index $\sigma : n \twoheadrightarrow k$ simply as $f(C_n)$ with the same index. Because we have the same index on the codomain, to show that $\Gamma(f)$ is a simplicial map, it's enough to look at the monomorphisms, for which the only interesting case is again d_n, which commutes, as f is a chain map. Rest of the functoriality is obvious. $\square$

We have a natural map $\Psi : \Gamma N(A) \longrightarrow A$ defined on the summand NA_k with the index $\sigma : n \twoheadrightarrow k$ as first embedding NA_k in A_k naturally and then pulling back by σ^*:

$$NA_k \hookrightarrow A_k \xrightarrow{\sigma^*} A_n$$

For naturality, we need to show that for a $\theta : m \longrightarrow n$ the following diagram commutes:

$$\begin{tikzcd}
\Gamma N(A)_n \arrow[r, "\Psi_n"] \arrow[d, "\theta^*"] & A_n \arrow[d, "\theta^*"] \\
\Gamma N(A)_m \arrow[r, "\Psi_m"] & A_m
\end{tikzcd}$$

If we focus on a single summand with the usual notation, we have:

$$\begin{tikzcd}
NA_k \arrow[r, hook] \arrow[d, "\phi^*"] & A_k \arrow[r, "\sigma^*"] \arrow[d, "\phi^*"] & A_n \arrow[d, "\theta^*"] \\
NA_s \arrow[r, hook] & A_s \arrow[r, "(\sigma')^*"] & A_m
\end{tikzcd}$$

Both squares commute clearly, where the second square is the induced square of the epi-mono factorization.

Theorem 1.8. *The map Ψ is a natural isomorphism from ΓN to Id_{sAb}.*

Proof. We just need to show that for each n, Ψ_n gives an isomorphism of abelian groups. This will be proven by induction on n.

For $n = 0$, there is only a single surjection going out of 0, which is $\mathrm{id} : 0 \longrightarrow 0$ and the statement becomes

$$\bigoplus_{0 \twoheadrightarrow 0} NA_0 = NA_0 \simeq A_0$$

through Ψ_n which is clear. Now assume the theorem for $k < n$. For surjectivity, by using Theorem 1.5 and the fact that NA_n is in the image through the index $\mathrm{id} : n \longrightarrow n$, we just need to show that for a $x \in DA_n$, we have x in the image, which gets reduced to showing that for every $y \in A_{n-1}$ and for every $j < n$, $s_j y \in A_n$. By the induction hypothesis, there is a $z \in \Gamma N(A)_{n-1}$ such that $\Psi_{n-1}(z) = y$ and let's say that $z \in NA_k$ with the index $\sigma : n-1 \twoheadrightarrow k$. Then for $z \in NA_k$ with the index σs^j we have $\Psi_n(z) = x$.

To show that Ψ_n is injection, we need to do more work. Suppose that for a family $(x_\sigma) \in \bigoplus_{\sigma: n \twoheadrightarrow k} NA_k$ we have

$$\Psi_n((x_\sigma)) = \sum_{\sigma: n \twoheadrightarrow k} \sigma^* x_\sigma = 0$$

Suppose that k as the domain of σ is greatest such that $x_\sigma \neq 0$. Observe that except $x_{n \longrightarrow n}$, every element of the family gets mapped to a degenerate simplex, so by Theorem 1.5 $x_{n \longrightarrow n}$ cannot be nonzero. This fact is proven in detail in the proof Theorem 1.9. We need further assumptions and ordering and for tha define a partial order on the set of surjections $n \twoheadrightarrow k$ by $\sigma_1 \leq \sigma_2$ if and only if $\sigma_1(j) \leq \sigma_2(j)$ for each $j \in n$. Now let $\sigma : n \twoheadrightarrow k$ be minimal. Observe that we are ordering surjections for which $x_\sigma \neq 0$. Furthermore $s_2 s_0 : 5 \longrightarrow 3$ and $s_1 s_1 : 5 \longrightarrow 3$ are not comparable, so in theory, there can be more than one minimal.

Now we want to construct a section ϕ of σ such that, if ϕ is a right inverse of another surjection σ', then $\sigma' \leq \sigma$. For this define

$$\begin{aligned} \phi : k &\longrightarrow n \\ j &\longmapsto \max \{l \in n \mid \sigma(l) = j\} \end{aligned}$$

With this definition, if we have additionally $\sigma' \psi = \mathrm{id}_k$, then if for $j < n$ we have $l_1 = \sigma'(j) > \sigma(j) = l_2$ and with $\psi(l_2) = t$, by the definition of ψ we get $t = \psi(l_2) \geq j$ however $l_2 = \sigma' \psi(l_2) = \sigma'(t) \geq \sigma'(j) = l_1$, which is a contradiction.

Since Ψ is a simplicial map, We have $\Psi_k d^*_{\Gamma N(A)} = d^*_A \Psi_n$, we have $\Psi_k \phi^* = \phi^* \Psi_n$. Because of the induction hypothesis, $\phi^*((x_\sigma)) = 0$. Now look at the square

$$\begin{tikzcd} k \arrow[r, "\psi"] \arrow[d, "\mathrm{id}"] & n \arrow[d, two heads, "\lambda"] \\ k \arrow[r, hook] & l_2 \end{tikzcd}$$

This square completely characterizes what is going to be in the image of ψ^* that corresponds to the index $id : k \longrightarrow k$. But for $l_2 > k$ the induced square applied to $x_{n \twoheadrightarrow l_2}$ gives zero. But the arrow at the bottom is an injection, so we must have

$$\begin{tikzcd} k \arrow[r, "\psi"] \arrow[d, "\mathrm{id}"] & n \arrow[d, two heads, "\lambda"] \\ k \arrow[r, "\mathrm{id}"] & k \end{tikzcd}$$

So, ψ is a section of λ. We have chosen σ in a way that it's minimal with $x_\sigma \neq$. But then, thanks to the way we constructed ψ, we have $\lambda \leq \sigma$, so x_λ

are all zero, which means in the image of ψ^*, x_σ (embedded) is solely what goes to the id : $k \longrightarrow k$. However, as we have seen before if Ψ_k takes an element of $\Gamma N(A)_k$ to 0, then that component must be zero, which gives a contradiction. $\square$

Now we are ready to prove the Dold-Kan correspondence:

Theorem 1.9 (Dold-Kan)**.** *The functors* $N : \mathbf{sAb} \longrightarrow \mathbf{Ch}$ *and* $\Gamma : \mathbf{Ch} \longrightarrow \mathbf{sAb}$ *form an equivalence of categories.*

Proof. In Theorem 1.8 we have proven the half of the theorem. Now we want to show that there is a natural isomorphism from $N\Gamma$ to $\mathrm{Id}_{\mathbf{Ch}}$. Let C be a chain complex and define $\Phi(C) : N\Gamma(C) \longrightarrow C$ as $\Phi_n(C) : (N\Gamma(C))_n \hookrightarrow (M\Gamma(C))_n = \Gamma(C)_n = \bigoplus_{n \twoheadrightarrow k} C_k$. It has been already discussed that this inclusion is natural, so it's enough to show that the image of this map is C_n.

First observe that $\mathrm{im}(\Phi_n(C)) = \bigcap_{i=0}^{n-1} \ker(d_i)$. Furthermore, $d^i : n-1 \hookrightarrow n \xrightarrow{\mathrm{id}} n$ has the epi-mono factorization of $n-1 \xrightarrow{\mathrm{id}} n-1 \overset{d^i}{\hookrightarrow} n$ and so for $i < n$ the induced map is the zero map on the component that's indexed by $\mathrm{id} : n \longrightarrow n$, which means that $C_n \subset \mathrm{im}(\Phi_n(C))$.

For the opposite direction, by Theorem 1.5, we have $\Gamma_n(C) \simeq (N\Gamma(C))_n \oplus (D\Gamma(C))_n$. Now we want to show that a component C_k for a $k < n$ with the index $\sigma : n \twoheadrightarrow k$ belongs to the degenerate part of this direct sum. Observe that such a surjection can be written as the composition $n \overset{s^i}{\twoheadrightarrow} n-1 \overset{\sigma'}{\twoheadrightarrow} k$ for an $i < n$ and then the component C_k with the index σ' in $\Gamma_{n-1}(C)$ hits C_k in $\Gamma_n(C)$ completely which can be seen by the diagram

$$\begin{tikzcd} n \arrow[r, two heads, "s^i"] \arrow[d, two heads, "\sigma"] & n-1 \arrow[d, two heads, "\sigma'"] \\ k \arrow[r, "\mathrm{id}"] & k \end{tikzcd}$$

Then we are done, because C_n is in the image and any other C_k is not. $\square$

1.2 Chain homotopy between NA and MA

Throughout this chapter, lower indices are for the usual indices of a chain map unless stated otherwise, for example when we define the chain homotopy. Also sometimes they're not denoted and clear from the context. The following theorem says that, in the eyes of homology, NA carries all the necessary information of a simplicial abelian group A.

Theorem 1.10. *The natural inclusion map* $i : NA \longrightarrow MA$, $A \in sAb$, *induces an isomorphism in the singular homology* $i_\star : H_\star(NA) \xrightarrow{\simeq} H_\star(MA)$.

Remark. *Chain homotopic maps induce the same homomorphism on singular homology. It's enough to show that i is a chain homotopy equivalence. A proof is spelled out in Proposition 2.12 of [5].*

Table 1 consists of homotopic chain complexes in each row, N_kA. First let i^k denote the inclusion $i^k : N_{k+1}A \longrightarrow N_kA$. Observe that $i_n = i_1^0 i_2^1 ... i_n^{n-1}$ and $i_0 = i_0^0 = \mathrm{id}_{A_0}$. This map goes upwards in the table. For the inverse direction (going downwards in the table), let $x \in N_kA_n$ and define

$$f^k(x) = \begin{cases} x & \text{for } n \leq k, \\ x - s_k d_k x & \text{for } n > k \end{cases}$$

Lemma 1.11. *f^k is a chain map from N_kA to $N_{k+1}A$ and the composite $f^k \circ i^k$ is the identity map $id_{N_{k+1}A}$.*

Proof. Main argument of showing that f^k is well-defined can be found in the proof of Theorem 1.5. In order to show that it's a chain map, there are two non-trivial cases. First,

$$\begin{array}{ccc} N_nA_{n+1} & \xrightarrow{\partial_{n+1}} & N_nA_n \\ \Big\downarrow{\scriptstyle \mathrm{id}-s_nd_n} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ N_{n+1}A_{n+1} & \xrightarrow{\partial_{n+1}} & N_{n+1}A_n \end{array}$$

Here $\partial_{n+1} = (-1)^{n+1}d_{n+1} + (-1)^n d_n$, so we want to show that $\partial_{n+1}x = \partial_{n+1}(x - s_nd_nx) = \partial_{n+1}x - \partial_{n+1}s_nd_nx$, which is equivalent to showing that $d_{n+1}s_nd_nx = d_ns_nd_nx$, and this follows by simplicial identity S3.

For the second case, assume $n > k+1$. Now we want to show that $\partial_n(x - s_kd_kx) = \partial_nx - s_kd_k\partial_nx$, so we need to show that $\partial_ns_kd_kx = s_kd_k\partial_nx$. But by the simplicial identities S3 we have the cancellation in the beginning and then by S4, we shift the indices:

$$\begin{aligned} \partial_n s_k d_k x &= \sum_{j=k+2}^{n} (-1)^j d_j s_k d_k x \\ &= \sum_{j=k+2}^{n} (-1)^j s_k d_{j-1} d_k x = \sum_{j=k+2}^{n} (-1)^j s_k d_k d_j x = s_k d_k \partial_n x \end{aligned}$$

Now, for the second part of the lemma, we just need to look at the case when $n > k+1$, but it's still trivial, because $s_kd_kx = s_k0 = 0$.

□

When we are going up in Table 1, we are embedding and when we're going down we are "retracting", the copy of $N_{k+1}A$ in N_kA stays unchanged under f^k. Now we want to define a homotopy between id_{N_kA} and $i^k \circ f^k$, which corresponds to going right in the table. Let $x \in N_kA_n$, and define $t_n^k : N_kA_n \longrightarrow N_kA_{n+1}$ by

$$t_n^k(x) = \begin{cases} 0 & \text{for } n < k, \\ (-1)^k s_k x & \text{for } n \geq k \end{cases}$$

One point to be careful about this notation is that when we substitute n in t_n^k, for example with $n-1$, then we substitute the n in the conditions too.

Lemma 1.12. *The chain map $i^k \circ f^k$ is homotopic to $\mathrm{id}_{N_k A}$ and this homotopy is given by t^k.*

Proof. Firstly, let $n \geq k$ and $x \in N_k A_n$. Then for $j < k-1$, we have $d_j(-1)^k s_k x = (-1)^k d_j s_k x = (-1)^k s_{k-1} d_j x = 0$, so t_n^k is well-defined.

Because of the definition of $N_k A_n$, the following diagram changes with different k and n. Combinatorial details of the proof depends on where we're in the Table 1.

$$\begin{tikzcd}
\dots \arrow[r] & N_kA_{n+1} \arrow[r] \arrow[d] & N_kA_n \arrow[r, "\partial_n"] \arrow[dl, "t_n^k"'] \arrow[d, shift right, "\mathrm{id}_{N_kA}"'] \arrow[d, shift left, "i^k\circ f^k"] & N_kA_{n-1} \arrow[r] \arrow[d] \arrow[dl, "t_{n-1}^k"] & \dots \\
\dots \arrow[r] & N_kA_{n+1} \arrow[r, "\partial_{n+1}"'] & N_kA_n \arrow[r] & N_kA_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

We want to show that

$$\partial_{n+1} t_n^k + t_{n-1}^k \partial_n = \mathrm{id}_{N_k A} - i^k \circ f^k \tag{2}$$

Let $x \in N_k A_n$. There are four cases:

Case 1: $n < k$: By definition $t_n^k(x) = t_{n-1}^k(x) = 0$ and $f^k(x) = x$.

Case 2: $n = k$: Again $t_{n-1}^k(x) = 0$ however $t_n^k(x) = (-1)_n^s x$ and $\partial_{n+1}(x) = (-1)^{n+1} d_{n+1} x + (-1)^n d_n x$. Using the simplicial identity S3, we have

$$(-1)^{n+1}(-1)^n d_{n+1} s_n x + (-1)^n(-1)^n d_n s_n x = d_n s_n x - d_{n+1} s_n x = x - x = 0$$

And $f^k(x) = x$, so the right-hand side of (2) becomes 0 too.

Case 3: $n = k+1$: Main observation for this case is that $\partial_{n+1}(x) = (-1)^{n+1} d_{n+1} x + (-1)^n d_n x + (-1)^{n-1} d_{n-1} x$. By using the simplicial identities S4 and S3 the first summand of the left-hand side of (2) becomes

$$\begin{aligned}
&(-1)^{n+1}(-1)^{n-1} d_{n+1} s_{n-1} x + (-1)^n (-1)^{n-1} d_n s_{n-1} x \\
&+ (-1)^{n-1}(-1)^{n-1} d_{n-1} s_{n-1} x = d_{n+1} s_{n-1} x - d_n s_{n-1} x + d_{n-1} s_{n-1} x \\
&= s_{n-1} d_n x - x + x = s_{n-1} d_n x
\end{aligned}$$

and the second summand becomes

$$(-1)^{n-1}(-1)^n s_{n-1} d_n x + (-1)^{n-1}(-1)^{n-1} s_{n-1} d_{n-1} x = s_{n-1} d_{n-1} x - s_{n-1} d_n x$$

So, the left-hand side is equal to $s_{n-1} d_{n-1} x$. We have also

$$\mathrm{id}_{N_k A}(x) - i^k \circ f^k(x) = x - (x - s_k d_k x) = s_{n-1} d_{n-1} x.$$

Case 4: $n > k+1$: Details of this case is same with the previous case, however the definitions for domains of the maps differ, which actually doesn't change any calculation.

In this case the equation (2) becomes

$$\sum_{j=k}^{n+1}(-1)^j(-1)^k d_j s_k x + \sum_{j=k}^{n}(-1)^k(-1)^j s_k d_j x = s_k d_k x$$

And by using the simplicial identities S3 and S4 we have

$$\begin{aligned}\sum_{j=k}^{n+1}(-1)^j(-1)^k d_j s_k x &= x - x + \sum_{j=k+2}^{n+1}(-1)^j(-1)^k d_j s_k x\\ &= \sum_{j=k+2}^{n+1}(-1)^j(-1)^k s_k d_{j-1} x = \sum_{j=k+1}^{n}(-1)^k(-1)^{j+1} s_k d_j x\end{aligned}$$

After the cancellation, we have the obvious equality. □

We want to use these row-by-row homotopies to construct a homotopy between NA and MA. For this, define $f : MA \longrightarrow NA$ by letting $f_n : MA_n \longrightarrow NA_n$ be the composite $f^{n-1}f^{n-2}...f^0$, and $f_0 = \mathrm{id}_{MA}$.

Proof of Theorem 1.10. It's clear that f is a well-defined chain map. Also Lemma 1.11 implies that $f \circ i = \mathrm{id}_{NA}$. For the reverse direction, define $T_n : A_n \longrightarrow A_{n+1} = t^0_n + i^0_{n+1}t^1_n f^0_n + ... + i^0_{n+1}...i^{n-1}_{n+1}t^n_n f^{n-1}_n...f^0_n$, so $T_0 = t^0_n$. We want to show that

$$\partial_{n+1}T_n + T_{n-1}\partial_n = \mathrm{id}_A - i_n \circ f_n = \mathrm{id}_A - i^0_n...i^{n-1}_n f^{n-1}_n...f^0_n$$

This might seem complicated, especially thanks to the indices. Lower indices are always completely clear from the domain naturally:

$$\begin{aligned}&\partial_{n+1}T_n + T_{n-1}\partial_n = \partial_{n+1}(t^0 + i^0t^1f^0 + ... + i^0...i^{n-1}t^nf^{n-1}...f^0)\\ &+ (t^0 + i^0t^1f^0 + ... + i^0...i^{n-2}t^{n-1}f^{n-2}...f^0)\partial_n\end{aligned} \tag{3}$$

Now we will use the fact that i^k and f^k are chain maps and (2) holds for every k and n. Observe that summands in (3) becomes, respectively

$$\partial_{n+1}t^0 + i^0\partial_{n+1}t^1f^0 + ... + i^0...i^{n-1}\partial_{n+1}t^nf^{n-1}...f^0$$

and

$$t^0\partial_n + i^0t^1\partial_nf^0 + ... + i^0...i^{n-2}t^{n-1}\partial_nf^{n-2}...f^0$$

Now, observe that for $j < n-1$ we have $\partial_{n+1}t^j + t^j\partial_n = \mathrm{id}_{N_jA} - i^j \circ f^j$. If we compose this equation with $i^0...i^{j-1}$ from left and with $f^{j-1}...f^0$ from right, we get

$$\begin{aligned}S_j &:= i^0...i^{j-1}\partial_{n+1}t^jf^{j-1}...f^0 + i^0...i^{j-1}t^j\partial_nf^{j-1}...f^0\\ &= i^0...i^{j-1}f^{j-1}...f^0 - i^0...i^jf^j...f^0\end{aligned}$$

with $S_0 = \mathrm{id}_A - i^0 \circ f^0$ Then, after putting the calculations together we get

$$\partial_{n+1} T_n + T_{n-1}\partial_n = \sum_{j=0}^{n-1} S_j + i^0 ... i^{n-1} \partial_{n+1} t^n f^{n-1} ... f^0$$

But observe that $\sum_{j=0}^{n-1} S_j = \mathrm{id}_A - i^0 ... i^{n-1} f^{n-1} ... f^0$, so now we just need to show that $i^0 ... i^{n-1} \partial_{n+1} t^n f^{n-1} ... f^0 = 0$. Let $x \in A_n$ and $y = f^{n-2} ... f^0(x)$. Then $t^n f^{n-1}(y) = (-1)^n s_n(y - s_{n-1} d_{n-1} y) \in N_n A_{n+1}$, so the differential ∂_{n+1} is just $(-1)^n d_n + (-1)^{n+1} d_{n+1}$. Now lastly by using the simplicial identities S3 we have

$$\begin{aligned} i^0 ... i^{n-1} \partial_{n+1} t^n f^{n-1} ... f^0(x) &= (-1)^n d_n (-1)^n s_n (y - s_{n-1} d_{n-1} y) \\ &+ (-1)^{n+1} d_{n+1} (-1)^n s_n (y - s_{n-1} d_{n-1} y) \\ &= (y - s_{n-1} d_{n-1} y) - (y - s_{n-1} d_{n-1} y) = 0 \end{aligned}$$

□

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